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A Law of the Iterated Logarithm for Distributions in the Generalized Domain of Attraction of a Nondegenerate Gaussian Law

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Summary. When operators T_n exist such that for sums S_n of n i.i.d. copies of a finite-dimensional random vector X we have $T_n S_n$ is shift-convergent in distribution to a standard Gaussian law, a necessary and sufficient condition on the distribution of X is given for the appropriate law of the iterated logarithm using the operators T_n to hold. Our result extends certain well-known real line L.I.L.'s; it utilizes a necessary and sufficient condition due to Hahn and Klass for T_n to exist giving a Gaussian limit law, and employs a second moment technique due to Kuelbs and Zinn.

1. Introduction

Let $X, X_1, X_2, \ldots,$ be independent, identically distributed (i.i.d.) random variables taking values in $\mathbb{R}^d$ equipped with the usual inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. Let $S=\{x\in\mathbb{R}^d\colon \|x\|=1\}$. When we wish to emphasize the role of $\theta\in S$ as a linear functional, we may write $\theta(x)$ for (θ, x). Put $S_n = X_1 + \ldots + X_n$, $n \geqq 1$.

When X is full (i.e., the distribution $\mathscr{L}(X)$ of X is not supported on any $(d-1)$ dimensional hyperplane), Hahn and Klass (1980a) established a necessary and sufficient condition for the existence of linear operators $\{T_n\}$ and shifts $\{b_n\}$ such that $\mathscr{L}(T_n(S_n - b_n)) \xrightarrow{w} \gamma$, where $\xrightarrow{w}$ denotes weak convergence and γ is the d-dimensional standard normal (i.e., Gaussian with identity covariance). Their condition is

$$\lim_{t\to\infty} \sup_{\theta\in S} \frac{t^2 P(|\theta(X)|>t)}{E(\theta^2(X)\wedge t^2)} = 0, \tag{1}$$

where $s \wedge t$ denotes the minimum of s and t.

When (1) holds, Hahn and Klass provide a construction of canonical $\{T_n\}$ which we will utilize extensively here. When $EX=0$, it is possible to take $b_n \equiv 0$. (It can be shown that (1) implies $E\|X\| < \infty$, so that EX exists.)

When $\mathscr{L}(T_n(S_n - b_n)) \xrightarrow{w} \gamma$ we say X belongs to the *generalized domain of* *attraction* of γ, and write $X \in \mathrm{GDA}(\gamma, \{T_n\}, \{b_n\})$. Operator normalization becomes necessary when different projections of X have different growth rates for their truncated variance functions. Examples were provided in Hahn and Klass' paper showing that the GDA of a full Gaussian is strictly larger than the (ordinary) domain of attraction obtained via normalizing constants (as opposed to operators).

It is clear that for full Gaussian limit laws it will suffice to consider the standard version γ, for if μ is Gaussian with mean vector b and nonsingular covariance operator Σ and $X \in \mathrm{GDA}(\mu, \{T_n\}, \{b_n\})$, then $X \in \mathrm{GDA}(\gamma, \{\Sigma^{-1/2} T_n\}, \{T_n^{-1} b + b_n\})$.

It is well known that central limit behavior of sums derived from X can imply strong limit theorems, in particular, laws of the iterated logarithm, for these sums. In the domain of attraction case (when $\mathscr{L}(S_n/d_n) \to \gamma$; here $EX=0$) in results of Kesten (1972), Klass (1976 and 1977) and Kuelbs and Zinn (1983) it was shown that almost-sure boundedness of $\{S_n/\gamma_n\}$ was equivalent to a certain moment-condition, namely $\sum_{n=1}^{\infty} P(\|X\| > \gamma_n) < \infty$, where we may take $\gamma_n = LLn\, d([n/LLn])$. Here, $Lx = \max(1, \log x)$, $L_2 x = LLx = L(Lx)$, and so on, and $[x]$ denotes the greatest integer in x.

Thus one conjectures that operator-normalization CLT behavior should imply a similar equivalence of the a.s. boundedness of the sequence $\{\Gamma_n S_n\}$, with $\Gamma_n = \frac{1}{LLn} T_{\left[\frac{n}{LLn}\right]}$, to the condition $\sum_{n=1}^{\infty} P(\|\Gamma_n X\| > 1) < \infty$. The "correctness" of these normalizations is demonstrated by proving, as in the constant normalizers case, that $\{\Gamma_n S_n\}$ has a nontrivial cluster set, and thus that $0 < \overline{\lim}_{n\to\infty} \|\Gamma_n S_n\| < \infty$, a.s. Here also, the cluster set ought to have nonempty interior in addition to being compact.

That this conjecture holds depends a great deal on the regularity of the normalizing sequence $\{\Gamma_n\}$, which in turn derives directly from the regularity of the CLT normalizers, $\{T_n\}$. This regularity allows us to work in blocks of length $n_k = 2^k$, where the normalizing operator is fixed, e.g. $\{\Gamma_{n_k} S_n : n_{k-1} < n \leqq n_k\}$. Thus when X is symmetric, standard inequalities such as Lévy's maximal and the Hoffmann-Jorgensen apply. In addition, the correct necessary condition also depends on regularity of $\{\Gamma_n\}$.

The required regularity of $\{T_n\}$ was proved in Hudson, Veeh and Weiner (1984) for general operator-stable limits. When applied here to the standard Gaussian case the result sharpens a great deal. In their Lemma 2, if we let $\mu = \gamma$ denote standard Gaussian measure on $\mathbb{R}^d$, then one exponent for γ is $B = \frac{1}{2} I$ (I being the identity on $\mathbb{R}^d$) so that $\beta^B = \sqrt{\beta}$. The group $S(\mu)$ is the familiar compact orthogonal group.

Our Theorem 1 is the main result, but as the proof is quite long, various technical lemmas and some straightforward but peripheral calculations have been deferred to the last section. However, the main proof is to be found in the next section. We note that the verification of the highly useful Kuelbs-Zinn

Doubly Truncated Second Moment Condition (see (17) and (20)ff.) forms the heart of the proof, and is, besides the regularity techniques, the main new feature of our proof.

The results in this paper form a portion of author's Ph.D. dissertation at the University of Wisconsin-Madison, under the direction of Professor James Kuelbs. The author wishes to acknowledge his indebtedness to Professor Kuelbs, and wishes also to thank Professor Alejandro de Acosta for many useful and stimulating conversations during this investigation.

2. The Main Theorem and Proof

Here is our main result. Recall $\Gamma_n = \frac{1}{LLn} T_{[\frac{n}{LLn}]}$.

Theorem 1. *Let $X \in \mathrm{GDA}(\gamma, \{T_n\}, \{b_n\})$, where γ is standard normal on $\mathbb{R}^d$. Then, there exists a finite constant C with*

$$\overline{\lim_{n\to\infty}} \|\Gamma_n(S_n - b_n)\| = C \quad \text{a.s.} \tag{2}$$

if and only if

$$\sum_n P(\|\Gamma_n X\| > \varepsilon) < \infty, \qquad \text{for some (for every) } \varepsilon > 0. \tag{3}$$

When (3) holds, the cluster set of $\{\Gamma_n(S_n - b_n)\}$ almost surely contains $\{x \in \mathbb{R}^d \colon \|x\| \leq \sqrt{2}\}$, and thus $\sqrt{2} \leq \overline{\lim_{n\to\infty}} \|\Gamma_n(S_n - b_n)\| = C < \infty$, a.s.

Proof. In order not to obscure the main outline of the proof, after carefully constructing a canonical version of Γ_n and describing the main regularity and moment tools we will need, we will prove the theorem directly, deferring the proofs of various technical lemmas until the end.

The proof of Hahn and Klass' Theorem (1980a) shows that when $X \in \mathrm{GDA}(\gamma, \{T_n\}, \{b_n\})$, EX exists and $b_n - nEX \to 0$. By replacing X by $X - EX$, we may therefore assume without loss of generality that $EX = 0$ and $b_n \equiv 0$.

It is clear that the statement of the theorem is independent of the choice of the normalizing sequence $\{T_n\}$ giving Gaussian convergence, for if $\mathscr{L}(U_n S_n) \xrightarrow{w} \gamma$ as well as $\mathscr{L}(T_n S_n) \xrightarrow{w} \gamma$, then $\{U_n T_n^{-1}\}$ is precompact with only orthogonal limits (by the Convergence of Types Theorem, Billingsley (1966)), since the group preserving γ is precisely the orthogonal group, so that

$$\overline{\lim} \left\| \frac{1}{LLn} U_{[\frac{n}{LLn}]} S_n \right\| = \overline{\lim} \left\| U_{[\frac{n}{LLn}]} T^{-1}_{[\frac{n}{LLn}]} \frac{1}{LLn} T_{[\frac{n}{LLn}]} S_n \right\|$$
$$= \overline{\lim} \left\| \frac{1}{LLn} T_{[\frac{1}{LLn}]} S_n \right\|.$$

Therefore we will always assume $\{T_n\}$ is the canonical Hahn-Klass sequence constructed below.

For $\theta \in S$ and $t>0$, define

$$U(\theta, t)=U_\theta(t)=E\theta^2(X)I(|\theta(X)|\leqq t),$$
$$M(\theta, t)=M_\theta(t)=E(\theta^2(X)\wedge t^2), \tag{4}$$

where I denotes indicator function.

Hahn and Klass prove that there exists u^* such that for each $t\geqq u^*$ there is a unique number $d(\theta, t)=d_\theta(t)$ satisfying

$$d_\theta^2(t)=t M_\theta(d_\theta(t)). \tag{5}$$

Moreover, the function $d\colon S\times[u^*, \infty)$ is jointly continuous and satisfies $\liminf_{t\to\infty}\inf_{\theta\in S} d(\theta, t)=\infty$. For each θ, $d(\theta, n)$ is the correct one-dimensional normalizing sequence for the sums $\theta(S_n)$, i.e., under (1) we have for each $\theta\in S$, $\mathscr{L}\left(\frac{\theta(S_n)}{d_\theta(n)}\right)\xrightarrow{w} N(0,1)$.

For $t\geqq u^*$ we construct an orthonormal basis for $\mathbb{R}^d$ as follows: since $d(\cdot, t)$ is continuous and S is compact, choose θ_{1t} satisfying $d(\theta_{1t}, t)=\inf_{\theta\in S} d(\theta, t)$. Inductively, having chosen θ_{jt} $(1\leqq j<k\leqq d)$, choose θ_{kt} so that

$$d(\theta_{kt}, t)=\inf\{d(\theta, t)\colon \theta\in S, \theta\perp\{\theta_{1t}, \ldots, \theta_{(k-1)t}\}\}.$$

Then $\{\theta_{jt}\colon 1\leqq j\leqq d\}$ is called a preferred o.n. basis (PONB) at time t.

The operators T_n accomplishing $\mathscr{L}(T_n S_n)\xrightarrow{w}\gamma$ are constructed by setting

$$T_t\theta_{jt}=\theta_{jt}/d(\theta_{jt}, t), \quad j=1, \ldots, d, \tag{6}$$

and extending T_t linearly to all of $\mathbb{R}^d$.

Note that T_t is well-defined even if t is not an integer.

There are two LIL-appropriate normalizers which we will demonstrate are equivalent, but at different times one is more convenient than the other. Certainly $\Gamma_n=\frac{1}{LLn}T_{\left[\frac{n}{LLn}\right]}$ will be equivalent to $\frac{1}{LLn}T_{\frac{n}{LLn}}$. The other useful possibility is to write (following Kuelbs and Zinn) $\alpha(t)=t/LLt$, noting that $\alpha^{-1}(t)$ is asymptotic to $tLLt$, and construct $\gamma\colon S\times[\alpha^{-1}(u^*), \infty)$ by writing $\gamma(\theta, t)=\gamma_\theta(t)=\alpha^{-1}(d_\theta(\alpha(t)))$. Then we define $\tilde{\Gamma}(\theta_{jt})=\theta_{jt}/\gamma(\theta_{jt}, t)$ $(j=1, \ldots, d)$, extended linearly. But from (21) below we will see that $\tilde{\Gamma}_n\Gamma_n^{-1}\to I$, so we may use the two interchangeably.

The utilization of the basic inequalities required in LIL proofs depends on regular behavior of the normalizing sequence. In Hudson, Veeh and Weiner (1984), Lemmas 2 through 4, the necessary regularity theorems were proved in the general setting of operator-stable convergence. We apply them here to the Gaussian case (where, in their notation, we have $S(\mu)$ is the orthogonal group of linear operators on $\mathbb{R}^d$, and $B=\frac{1}{2}I$ is an exponent of μ, revealing $\beta^B=\sqrt{\beta}$). We refer the reader there for the proofs and details. Recall that $\Gamma_n=\frac{1}{LLn}T_{\left[\frac{n}{LLn}\right]}$, where LLn and $\left[\frac{n}{LLn}\right]$ each vary regularly. In what follows let γ denote standard Gaussian measure on $\mathbb{R}^d$.

Lemma 2 (Regularity for Gaussian Convergence). *Suppose* $\mathscr{L}(T_n S_n) \xrightarrow{w} \gamma$. *Let* $I(K) = \{n \in \mathbb{N}: n_{K-1} < n \leqq n_K\}$, *where* $n_K = 2^K$. *Then*

(a) $\lim_{n\to\infty} \|\Gamma_n \Gamma_{n-1}^{-1}\| = 1$.

(b) *If* $\{c_n\}$, $\{d_n\} \subset \mathbb{N}$ *with* $c_n \to \infty$, $d_n \to \infty$ *and* $d_n/c_n \to c \in [0, \infty)$, *then* $\lim_{n\to\infty} \|\Gamma_{c_n} \Gamma_{d_n}^{-1}\| = \sqrt{c}$.

(c) $\lim_{K\to\infty} \max_{n\in I(K)} \|\Gamma_n \Gamma_{n_K}^{-1}\| = \sqrt{2}$, $\lim_{K\to\infty} \min_{n\in I(K)} \|\Gamma_{n_K} \Gamma_n^{-1}\| = 1/\sqrt{2}$.

(d) *If* $\{c_n\}, \{d_n\} \subset \mathbb{N}$ *with* $c_n \to \infty$ *and* $d_n/c_n \to \infty$, *then* $\lim_{n\to\infty} \frac{1}{\log(d_n/c_n)} \cdot \log \|T_{c_n} T_{d_n}^{-1}\| = 1/2$.

We remark that (d) implies $\Gamma_n S_n \xrightarrow{p} 0$ ($\xrightarrow{p}$ denotes convergence in probability), since with $c_n = [n/LLn]$ and $d_n = n$ we have for $0 < \delta < \frac{1}{2}$,

$$\|\Gamma_n S_n\| \leqq \|\Gamma_n T_n^{-1}\| \, \|T_n S_n\| = \frac{1}{LLn} \|T_{\left[\frac{n}{LLn}\right]} T_n^{-1}\| \, \|T_n S_n\| \leqq \frac{(LLn)^{\frac{1}{2}+\delta}}{LLn} \|T_n S_n\| \xrightarrow{p} 0.$$

This fact will be crucial in applying Hoffmann-Jorgensen's inequalities and in the proof of clustering.

We will require several equivalent formulations of condition (3), proof for which we defer to Sect. 3. The technique is standard (cf. Stout (1974)) once Lemma 2 is known.

Lemma 3 (Moment Equivalences). *When* $\mathscr{L}(T_n S_n) \xrightarrow{w} \gamma$ *and* $\Gamma_n = \frac{1}{LLn} T_{\left[\frac{n}{LLn}\right]}$, *the following are equivalent:*

(a) $\sum_n P(\|\Gamma_n X\| > \varepsilon) < \infty$, *for some* $\varepsilon > 0$

(b) $\sum_n P(\|\Gamma_n X\| > \varepsilon) < \infty$, *for every* $\varepsilon > 0$

(c) $\int_{u^*}^{\infty} \sup_{\theta\in S} P(|\theta(X)| \geqq \gamma(\theta, t))\, dt < \infty$ (7)

(d) *For some* $A < \infty$, $\overline{\lim}_n \|\Gamma_n X_n\| \leqq A$, a.s.

(e) $\Gamma_n X_n \to 0$ a.s.

(f) *With* $n_K = 2^K$, $\sum_K n_K P(\|\Gamma_{n_K} X\| > \varepsilon) < \infty$ *for every* $\varepsilon > 0$.

We remark that (c) will be used in the crucial verification of a series condition whose sufficiency for our purposes derives from the technique of Kuelbs and Zinn (1979), and which verification is the main new technique of our proof.

To prove the necessity of (3) for (2), observe that Lemma 2(a) shows that (2) implies $\overline{\lim}_{n\to\infty} \|\Gamma_n X_n\| \leqq \overline{\lim}_n \|\Gamma_n S_n\| + \overline{\lim}_n \|\Gamma_n S_{n-1}\|$ (by the triangle inequality) $\leqq C + (\lim_n \|\Gamma_n \Gamma_{n-1}^{-1}\|) \overline{\lim}_{n\to\infty} \|\Gamma_{n-1} S_{n-1}\| \leqq 2C$, a.s., so that Lemma 3 ((d) implies (a)) shows (3) holds.

The proof of the sufficiency of (3) for (2) is achieved using the regularity lemma, classical truncation levels adapted to the operator case, and several by now standard inequalities and tools developed in past *LIL* proofs. New ingredients include the technique for final elimination of the middle part of the sums, which depends on using the uniform regularity of the functions $\gamma(\theta, t)$ to verify a sufficient condition due to Kuelbs and Zinn (1979). Previous verifications of this condition in related situations (see Goodman, Kuelbs and Zinn (1981); Kuelbs and Zinn (1983), for example) have involved a counting argument and reversal of summation which is rather inappropriate in the operator setting.

In what follows let $n_K = n(K) = 2^K$, and denote by $I(K)$ the block of integers $\{n: n(K-1) < n \leq n(K)\}$. *We assume that X is symmetric;* we will show later that desymmetrization can be arranged in a manner similar to that of Kuelbs and Zinn (1983).

For $K \geq 1$ and $j \leq n_K$, put

$$\begin{aligned} u_j &= u_j(K) = X_j I(\|\Gamma_{n_K} X_j\| \leq 1/LLn_K) \\ v_j &= v_j(K) = X_j I\left(\frac{1}{LLn_K} < \|\Gamma_{n_K} X_j\| \leq 1\right) \\ w_j &= w_j(K) = X_j I(\|\Gamma_{n_K} X_j\| > 1). \end{aligned} \tag{8}$$

In what follows, repeated use of the regularity lemma will allow us to work in the blocks $I(K)$ where the normalizing operators are held fixed, so that familiar maximal inequalities can be applied.

Note

$$\begin{aligned} \max_{n \in I(K)} \|\Gamma_n S_n\| \leq & \max_{n \in I(K)} \|\Gamma_n \Gamma_{n_K}^{-1}\| \, \|\Gamma_{n_K}(S_n - S_{n_{K-1}})\| \\ & + (\max_{n \in I(K)} \|\Gamma_n \Gamma_{n_{K-1}}^{-1}\|) \, \|\Gamma_{n_{K-1}} S_{n_{K-1}}\|. \end{aligned} \tag{9}$$

Now the regularity lemma gives $\max_{n \in I(K)} \|\Gamma_n \Gamma_{n_K}^{-1}\| \to \sqrt{2}$ as $K \to \infty$, while $\max_{n \in I(K)} \|\Gamma_n \Gamma_{n_{K-1}}^{-1}\| \to 1$. But symmetry and Lévy's inequality imply, for any $t > 0$,

$$P(\max_{n \in I(K)} \|\Gamma_{n_K}(S_n - S_{n_{K-1}})\| > t) \leq 2P(\|\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}})\| > t).$$

Thus, if we can show that for some $t > 0$,

$$\sum_K P(\|\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}})\| > t) < \infty, \tag{10}$$

we will obtain $\overline{\lim_n} \|\Gamma_n S_n\| \leq \sqrt{2}\, t + \overline{\lim_k} \|\Gamma_{n_{K-1}} S_{n_{K-1}}\|$, a.s. via the Borel-Cantelli lemma, along with $\overline{\lim_K} \|\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}})\| \leq t$, a.s.

But the identity

$$\Gamma_{n_K} S_{n_K} = \sum_{j=1}^{K} (\Gamma_{n_K} \Gamma_{n_j}^{-1}) \Gamma_{n_j}(S_{n_j} - S_{n_{j-1}})$$

(with $S_{n_0}=0$), together with Lemma 7 in Sect. 3, shows that $\overline{\lim}_K \|\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}})\| \leqq t$ a.s. implies $\overline{\lim}_K \|\Gamma_{n_K} S_{n_K}\| \leqq \frac{\sqrt{2}}{\sqrt{2}-1} t$ a.s. Thus, (9) and the verification of (10) will give us

$$\overline{\lim}_n \|\Gamma_n S_n\| \leqq t\left(\sqrt{2}+\frac{\sqrt{2}}{\sqrt{2}-1}\right), \quad \text{a.s.}$$

Now, because of the triangle inequality, to verify (10) it suffices to show there exist $A>0$, $B>0$ and $C>0$ such that

$$\sum_K P(\|\Gamma_{n_K} \sum_{j\in I(K)} u_j(K)\| > A) < \infty \tag{11}$$

$$\sum_K \| P(\|\Gamma_{n_K} \sum_{j\in I(K)} v_j(K)\| > B) < \infty \tag{12}$$

and

$$\sum_K P(\|\Gamma_{n_K} \sum_{j\in I(K)} w_j(K)\| > C) < \infty. \tag{13}$$

The easiest of these, (13), holds for any $C>0$ due to (3), Lemma 3 (f), and

$$\sum_K P(\|\Gamma_{n_K} \sum_{j\in I(K)} w_j(K)\| > C) \leqq \sum_K n_{K-1} P(\|\Gamma_{n_K} X\| > 1) < \infty,$$

so that in fact we have $\Gamma_{n_K} \sum_{j\in I(K)} w_j \to 0$, a.s.

Rather than verifying (11), we will instead show

$$\sum_K P\left(\left\|\Gamma_{n_K} \sum_{j=1}^{n_K} u_j(K)\right\| > A\right) < \infty, \tag{11'}$$

which will imply (11) because of Levy's inequality.

The following argument adapts to our situation a new method introduced by Kuelbs and Ledoux (1984). We remark that this method improves on the earlier technique of Pisier (1975) in dealing with the "small" random variables in the double truncation scheme usually employed.

To verify (11'), we will utilize an exponential inequality found in de Acosta (1980), which we restate here for reference.

Lemma 4 (de Acosta (1980)). *Let $\{Y_j: j=1, \ldots, n\}$ be an independent sequence of symmetric random vectors such that $\|Y_j\| \leqq c$, and put $Z_n = \sum_{j=1}^n Y_j$. Then, for each $\lambda>0$ and $t>0$,*

$$E\exp(\lambda\|Z_n\|) \leqq \frac{e^{\lambda t}}{1-2e^{\lambda(t+c)}P(\|Z_n\|>t)}. \tag{14}$$

Before applying Lemma 4, let us introduce the following notation:

$$m_K = [n_K/LLn_K], \qquad p_K = [n_K/m_K],$$

$$A_K = T_{m_K} = (LLn_K)\Gamma_{n_K}, \quad V_s^t = \sum_{j=s}^{t} u_j(K).$$

We observe that $p_K \sim LLn_K$, $p_K m_K \sim n_K$, so that the regularity lemma and Lévy's inequality give tightness of

$$\{\Lambda_K V_{jm_K}^{(j+1)m_K}: j \leqq p_K, K \geqq 1\} \cup \{\Lambda_K V_{p_K m_K}^{n_K}: K \geqq 1\}.$$

Choose $t > 1$ so large that, uniformly in K,

$$\max(\max_{j \leqq p_K} P(\|\Lambda_K V_{jm_K}^{(j+1)m_K}\| > t), P(\|\Lambda_K V_{p_K m_K}^{n_K}\| > t)) \leqq e^{-2}/3. \tag{15}$$

Then, letting $\lambda > 0$ be a constant to be determined later, we have

$$\begin{aligned} P\left(\|\Gamma_{n_K} \sum_{j=1}^{n_K} u_j\| > A\right) &= P\left(\left\|\Lambda_K \sum_{j=1}^{n_K} u_j\right\| > ALLn_K\right) \\ &\leqq \exp(-\lambda ALLn_K) E \exp\left(\lambda \left\|\Lambda_K \sum_{j=1}^{n_K} u_j\right\|\right) \\ &\leqq \exp(-\lambda ALLn_K) \prod_{j=1}^{p_K} E \exp(\lambda \|\Lambda_K V_{jm_K}^{(j+1)m_K}\|) \\ &\quad \times E \exp(\lambda \|\Lambda_K V_{P_K m_K}^{n_K}\|) \\ &= \exp(-\lambda ALLn_K)(E \exp(\lambda \|\Lambda_K V_0^{m_K}\|))^{P_K} E \exp(\lambda \|\Lambda_K V_{P_K m_K}^{n_K}\|). \end{aligned} \tag{16}$$

Using (15) and Lemma 4 we can estimate the exponential moments above: Notice $\|\Lambda_K u_j(K)\| \leqq 1$ $(j \leqq n_K)$, so in the lemma take $c=1$, $Y_j = \Lambda_K u_j$, to obtain

$$a_K = \max(E \exp(\lambda \|\Lambda_K V_0^{m_K}\|), E \exp(\lambda \| V_{P_K m_K}^{n_K}\|)) \leqq e^{\lambda t}\left(1 + \frac{2e^{\lambda(t+1)} e^{-2}/3}{1 - 2e^{\lambda(t+1)} e^{-2}/3}\right).$$

Now let $\lambda = 1/t$, and use the inequality $1 + x \leqq e^x$ to obtain from (16) (recalling $t > 1$) for large K,

$$\begin{aligned} P\left(\left\|\Gamma_{n_K} \sum_{j=1}^{n_K} u_j\right\| > ALLn_K\right) &\leqq \exp\left(\frac{-A}{t} LLn_K\right) a_K^{P_K+1} \\ &\leqq \exp\left(LLn_K\left(\frac{-A}{t} + 1 + 2\right)\right), \end{aligned}$$

which is summable over K provided $A > 4t$.

Thus (11′) and hence (11) hold for some $A > 0$; we remark that the argument above can be tightened slightly to give an estimate of the best choice of A above, as in, for example, Kuelbs and Zinn (1983). However, even in the constants case the estimate obtained in this way is far from the true value of $\sqrt{2}$ on the line implicit in Klass (1977, p. 154). In Kuelbs (1984) (again, using constant normalizations in Banach space) the correct value of $\sqrt{2}$ is obtained using linear functionals and the real result, but in the case of operator normalizations this approach does not work, at least in a natural way, since if $\theta \in S$, then $(\theta, \Gamma_n S_n)$ is not a normalized i.i.d. sum derived from *one* real r.v., but rather a mixture, and so the correct value of $\overline{\lim}_n |(\theta, \Gamma_n S_n)|$ us not at all obvious from the real line result. Thus we will omit any upper estimate of $\overline{\lim_n} \|\Gamma_n S_n\|$ (aside from its finiteness), although we will provide the lower estimate of $\sqrt{2}$ later on.

Finally, that the value of $\overline{\lim_n} \|\Gamma_n S_n\|$ is actually a constant almost surely (i.e., is nonrandom) follows from the zero-one law, which applies since $\lim_{n\to\infty} \sup_{\|\theta\|=1} \|\Gamma_n \theta\| = 0$ so that for any N, $\lim_{n\to\infty} \Gamma_n S_N = 0$ a.s.

The verification of (12), which is the heart of our proof, depends on a sufficient condition developed by Kuelbs and Zinn (1979). We will adapt their technique to the operators case, and then proceed to verify their condition (let us refer to it as the Kuelbs-Zinn Doubly Truncated Second Moment Condition, or KZDTSMC) using a strong uniform regularity property of the functions $\gamma(\theta, t)$ which we will establish. We remark that double truncation itself was introduced by Erdös.

Let $L_K = n_{K-1} E\|\Gamma_{n_K} X\|^2 I\left(\frac{1}{LLn_K} < \|\Gamma_{n_K} X\| \leqq 1\right)$. We assert that to verify (12) (still assuming (3)) it suffices to show that for some $r > 0$ we have the

$$\text{(KZDTSMC)} \quad \sum_K L_K^r < \infty. \tag{17}$$

To see that (17) implies (12) we use a device introduced by Kuelbs and Ledoux (1984). Observe that $\|\Gamma_{n_K} v_j\| \leqq 1$ for $j \in I(K)$, so that $P(\max_{j\in I(K)} \|\Gamma_{n_K} v_j\| > \lambda) = 0$ for $\lambda > 1$.

With r as in (17), let m be an integer large enough that $2^m \geqq r$. Choose B so large that $B > 3^m$, and then put $\lambda = B3^{-m}$, so that $\lambda > 1$.

Iterating Hoffmann-Jorgensen's inequality m times yields

$$\begin{aligned} P(\|\Gamma_{n_K} \sum_{j\in I(K)} v_j(K)\| > B) &= P(\|\Gamma_{n_K} \sum_{j\in I(K)} v_j\| > 3^m \lambda) \\ &\leqq (\text{const})\, P^{2^m}(\|\Gamma_{n_K} \sum_{j\in I(K)} v_j\| > \lambda) \\ &\leqq (\text{const})\, P^r(\|\Gamma_{n_K} \sum_{j\in I(K)} v_j\| > \lambda). \end{aligned} \tag{18}$$

But independence and symmetry give (since $\|\cdot\|$ is Euclidean)

$$\begin{aligned} E\|\Gamma_{n_K} \sum_{j\in I(K)} v_j\|^2 &= \sum_{j\in I(K)} E\|\Gamma_{n_K} v_j\|^2 \\ &= n_{K-1} E\|\Gamma_{n_K} X\|^2 I\left(\frac{1}{LLn_K} < \|\Gamma_{n_K} X\| \leqq 1\right), \end{aligned}$$

and hence by Markov's inequality,

$$P(\|\Gamma_{n_K} \sum_{j\in I(K)} v_j\| > \lambda) \leqq \frac{1}{\lambda^2} L_K. \tag{19}$$

Combining (18), (19) and (17) gives us (12).

The standard verifications of the KZDTSMC (17), as appearing in Goodman, Kuelbs and Zinn (1981) and Kuelbs and Zinn (1983), are applied to the case $r=2$ using a counting/summation-reversal argument; we were unable to apply it here. Instead, we verify (17) for any $r>3$, and as any $r>0$ would suffice, our proof of $\overline{\lim_n}\,\|\Gamma_n S_n\|<\infty$ in the symmetric case will be complete.

First, we claim we can replace (17) by the condition in $\mathbb{R}^d$

$$\sum_K \left(\frac{1}{LLn_k}\sup_{\theta\in S}\frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{M_\theta(\gamma_\theta(n_K)/LLn_K)}\right)^r<\infty. \tag{20}$$

Before showing that (20) actually implies the KZDTSMC (17), we remark that although the proof until now has been worked almost exclusively using operators without utilization of the Hahn-Klass canonical form (5) and (6), the remainder will be worked directly in terms of functionals, starting from condition (7) instead of (3), and depends on uniform regularity of $\{\gamma_\theta(t)\colon \theta\in S\}$ rather than regularity of $\{T_n\colon n\geqq 1\}$. Thus condition (1), which allows these uniformity lemmas to be proved, now becomes essential to the argument.

The key identity to keep in mind is this consequence of (5):

$$\gamma_\theta^2(t)=(\alpha^{-1}d_\theta\alpha(t))^2\approx t(LLt)M_\theta(\gamma_\theta(t)/LLt), \tag{21}$$

where the notation $a_\theta(t)\approx b_\theta(t)$ means $\limsup_{t\to\infty}\sup_{\theta\in S}\left|\frac{a_\theta(t)}{b_\theta(t)}-1\right|=0$. We know that $d_\theta(t)$ grows uniformly at least as fast as $t^{1/2}$, but Lemma 5 in Hahn and Klass (1980b) (see next paragraph) shows that for each $\delta>0$, we have $\lim_{t\to\infty}\sup_{\theta\in S}\frac{d_\theta(t)}{t^{\frac{1}{2}+\delta}}=0$. Thus $LLd_\theta(t)\approx LLt$, and (21) holds.

To see that (20) implies the KZDTSMC (17), we recall the canonical definitions of Γ_n and $\tilde{\Gamma}_n$. It is clear that since Γ_n, $\tilde{\Gamma}_n$ are each diagonal with respect to the same basis (the PONB at time n/LLn) with entries $LLnd_\theta(n/LLn)$ versus $\gamma_\theta(n)$, we see that (21) and Lemma 3 allow us to assume that v_j was defined in (8) with respect to $\tilde{\Gamma}_n$ rather than Γ_n, our error involving at most a constant factor introduced on the left hand side of (17) and (20). For, Lemma 5 in Hahn and Klass (1980b) and (21) provide that $\lim_{t\to\infty}\sup_{\theta\in S}\left|\frac{\gamma_\theta(ct)}{\gamma_\theta(t)}-c^{1/2}\right|=0$, since they prove the corresponding statement with d_θ replacing γ_θ, for each $c>0$.

Thus, we are asked to show that (20) implies that

$$\sum_K\left\{n_K E\|\tilde{\Gamma}_{n_K}X\|^2 I\left(\frac{1}{LLn_K}<\|\tilde{\Gamma}_{n_K}X\|\leqq 1\right)\right\}^r<\infty. \tag{22}$$

To ease the notation we just write Γ_n for $\tilde{\Gamma}_n$, and only give the proof in the case d (the dimension of our space) is two, for the general case follows exactly the same way.

On the event $\|\Gamma_{n_K}X\|\in\left(\frac{1}{LLn_K}, 1\right]$, we have

$$\frac{1}{(LLn_K)^2}<\|\Gamma_{n_K}X\|^2=\frac{\theta_1(X)^2}{\gamma_{\theta_1}(n_K)^2}+\frac{\theta_2(X)^2}{\gamma_{\theta_2}(n_K)^2}\leqq 1, \tag{23}$$

where we let $\{\theta_1, \theta_2\}=\{\theta_1(K), \theta_2(K)\}$ denote the PONB at time n_K/LLn_K.

But (23) implies $\frac{|\theta_i(X)|}{\gamma_{\theta_i}(n_k)}\leqq 1$ $(i=1,2)$ and also that at least for one i we have $\frac{|\theta_i(X)|}{\gamma_{\theta_i}(n_k)}>\frac{1}{2LLn_k}$. Therefore

$$\begin{aligned}
&E\frac{\theta_1(X)^2}{\gamma_{\theta_1}(n_k)^2}I\left(\|\Gamma_{n_k}X\|\in\left(\frac{1}{LLn_k}, 1\right]\right)\\
&\quad\leqq E\frac{\theta_1(X)^2}{\gamma_{\theta_1}(n_k)^2}I\left(\frac{|\theta_1(X)|}{\gamma_{\theta_1}(n_k)}\in\left(\frac{1}{2LLn_k}, 1\right]\right)\\
&\quad\quad+E\frac{\theta_1(X)^2}{\gamma_{\theta_1}(n_k)^2}I\left(\frac{|\theta_1(X)|}{\gamma_{\theta_1}(n_k)}\leqq\frac{1}{2LLn_k}, \frac{|\theta_2(X)|}{\gamma_{\theta_2}(n_k)}\in\left(\frac{1}{2LLn_k}, 1\right]\right)\\
&\quad\leqq\sup_{\theta\in S}E\frac{\theta^2(X)}{\gamma_\theta(n_k)^2}I\left(\frac{|\theta(X)|}{\gamma_\theta(n_k)}\in\left(\frac{1}{2LLn_k}, 1\right]\right)\\
&\quad\quad+\frac{1}{4(LLn_k)^2}P\left(\frac{|\theta_2(X)|}{\gamma_{\theta_2}(n_k)}\in\left(\frac{1}{2LLn_k}, 1\right]\right).
\end{aligned}\tag{24}$$

But

$$\begin{aligned}
P\left(\frac{|\theta_2(X)|}{\gamma_{\theta_2}(n_K)}\in\left(\frac{1}{2LLn_K}, 1\right]\right)&\leqq 4(LLn_K)^2E\frac{\theta_2(X)^2}{\gamma_{\theta_2}(n_K)^2}I\left(\frac{|\theta_2(X)|}{\gamma_{\theta_2}(n_K)}\in\left(\frac{1}{2LLn_K}, 1\right]\right)\\
&\qquad\text{(by Markov's inequality)}\\
&\leqq 4(LLn_K)^2\sup_{\theta\in S}E\frac{\theta^2(X)}{\gamma_\theta(n_K)^2}I\left(\frac{|\theta(X)|}{\gamma_\theta(n_K)}\in\left(\frac{1}{2LLn_K}, 1\right]\right),
\end{aligned}$$

and so (23), (24), and the same argument of (24) applied to θ_2 instead of θ_1, lead to

$$\begin{aligned}
&E\|\Gamma_{n_K}X\|^2I\left(\|\Gamma_{n_K}X\|\in\left(\frac{1}{LLn_K}, 1\right]\right)\\
&\qquad\leqq 4\sup_{\theta\in S}E\frac{\theta^2(X)}{\gamma_\theta(n_K)^2}I\left(\frac{|\theta(X)|}{\gamma_\theta(n_K)}\in\left(\frac{1}{2LLn_K}, 1\right]\right).
\end{aligned}\tag{25}$$

Now (21) shows that $\gamma_\theta(n_K)^2\approx d_\theta^2(n_K/LLn_K)LLn_K$, so the right hand side of (25) is dominated by some constant times

$$\frac{1}{n_KLLn_K}\sup_{\theta\in S}\frac{E\theta^2(X)I\left(\frac{|\theta(X)|}{\gamma_\theta(n_K)}\in\left(\frac{1}{2LLn_K}, 1\right]\right)}{M_\theta(d_\theta(n_K/LLn_K))}, \tag{26}$$

by using (4) and (5).

But of course,

$$E\theta^2(X)I\left(\frac{|\theta(X)|}{\gamma_\theta(n_k)}\in\left(\frac{1}{2LLn_K},1\right]\right)=U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/2LLn_K).$$

Thus, (20) implies (17), as desired.

To verify (20) we recall that by Lemma 3, (3) implies (7),

$$\int_1^\infty \sup_{\theta\in S} P(|\theta(X)|>\gamma_\theta(t))\,dt<\infty.$$

Fix $r>3$ and choose $\delta>0$ so that $r>\frac{3}{1-\delta}$. Now from Lemma 8 in the last section we see that

$$\limsup_{K\to\infty}\sup_{\theta\in S}\frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{(LLn_K)^\delta M(\gamma_\theta(n_K)/LLn_K)}$$

$$\leqq \limsup_{K\to\infty}\sup_{\theta\in S}\frac{U_\theta(\gamma_\theta(n_K))}{(LLn_K)^\delta U_\theta(\gamma_\theta(n_K)/LLn_K)}=0.$$

A closer look at the functions $\gamma_\theta(t)$ is required. The definition of $\gamma_\theta(t)$ implies that $\gamma_\theta(n)$ is a suitable normalization for $\theta(S_n)$; indeed the real-line LIL (e.g., Feller (1968)) and Lemma 3(c) imply $\overline{\lim}_{n\to\infty}\frac{|\theta(S_n)|}{\gamma_\theta(n)}=\sqrt{2}$ a.s. Now (4) and (5) amount to saying that γ_θ is the inverse function of $\gamma_\theta^{-1}(t)=\alpha^{-1}g_\theta\alpha(t)$, where $g_\theta(t)=t^2/M_\theta(t)$. It is well-known (see, e.g., Hahn and Klass (1980a)) that for sufficiently large t, uniformly in $\theta\in S$, we have $\gamma_\theta^{-1}(t)$ well-defined and continuous (in both θ and t). Bearing in mind (21), note that for s tending to ∞,

$$\gamma_\theta^{-1}(s)\approx LLs\,g_\theta(s/LLs)=(LLs)\frac{s^2}{(LLs)^2}\frac{1}{M_\theta(s/LLs)}=\frac{s^2}{LLs\,M_\theta(s/LLs)}. \tag{27}$$

Putting $s=\gamma_\theta(n_K)/dLLn_K$ in (27) shows there exists a positive constant C_1 such that, uniformly in $\theta\in S$, for all sufficiently large K, we have (see (21))

$$\gamma_\theta^{-1}(\gamma_\theta(n_K)/dLLn_K)\geqq\frac{C_1 n_K M_\theta(\gamma_\theta(n_K)/LLn_K)}{(LLn_K)^2 M_\theta(\gamma_\theta(n_K)/(LLn_K)^2)}\geqq C_1\frac{n_K}{(LLn_K)^2}. \tag{28}$$

Inequality (28) is vital in proving (20) (hence (17), the KZDTSMC). To use it we let $m_K=C_1 n_k/(LLn_K)^2$. Then (28) becomes

$$\gamma_\theta(n_K)/dLLn_K\geqq\gamma_\theta(m_K); \tag{29}$$

note that $m_K\nearrow\infty$ strictly, and $\frac{m_K-m_{K-1}}{m_{K-1}}\to 1$.

In the following $C_2, C_3, C_4, \ldots,$ are unimportant positive constants required to use the asymptotic statements mentioned above.

Recalling (7), we have (since $\gamma_\theta(t)\nearrow$)

$$\begin{aligned}\infty &> \int_1^\infty \sup_{\theta\in S} P(|\theta(X)|>\gamma_\theta(t))\,dt\\ &\geqq C_2 \sum_{K=1}^\infty \int_{m_{K-1}}^{m_K} \sup_{\theta\in S} P(|\theta(X)|>\gamma_\theta(t))\,dt\\ &\geqq C_2 \sum_{K=1}^\infty (m_K-m_{K-1}) \sup_{\theta\in S} P(|\theta(x)|>\gamma_\theta(m_K))\\ &\geqq C_3 \sum_{K=1}^\infty m_K \sup_{\theta\in S} P(\gamma_\theta(n_K)\geqq|\theta(X)|>\gamma_\theta(m_K))\\ &\geqq C_3 \sum_{K=1}^\infty m_K \sup_{\theta\in S} P(\gamma_\theta(n_K)\geqq|\theta(X)|>\gamma_\theta(n_K)/dLLn_K), \quad \text{by (29).}\end{aligned} \tag{30}$$

Let F_θ denote the cumulative distribution function of $|\theta(X)|$. Then (30) gives, recalling (21),

$$\begin{aligned}\infty &> \sum_{K=1}^\infty \frac{n_K}{(LLn_K)^2} \sup_{\theta\in S} \frac{1}{\gamma_\theta(n_K)^2} \int_{\gamma_\theta(n_K)/dLLn_K}^{\gamma_\theta(n_K)} x^2\,dF_\theta(x)\\ &\geqq C_4 \sum_{K=1}^\infty \frac{1}{(LLn_K)^3} \sup_{\theta\in S} \frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{M_\theta(\gamma_\theta(n_K)/LLn_K)}\\ &= C_4 \sum_{k=1}^\infty \frac{1}{(LLn_K)^{3-\delta}} \left\{\sup_{\theta\in S} \frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{(LLn_K)^\delta M_\theta(\gamma_\theta(n_K)/LLn_K)}\right\}\\ &\geqq C_5 \sum_{K=1}^\infty \frac{1}{(LLn_K)^{3-\delta}} \left\{\sup_{\theta\in S} \frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{(LLn_K)^\delta M_\theta(\gamma_\theta(n_K)/LLn_K)}\right\}^r\\ &\qquad\qquad \text{(using the fact that } \{\cdot\}\to 0)\\ &= C_5 \sum_{K=1}^\infty \frac{1}{(LLn_K)^{3-\delta+r\delta}} \left\{\sup_{\theta\in S} \frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{M_\theta(\gamma_\theta(n_K)/LLn_K)}\right\}^r\\ &\geqq C_6 \sum_{K=1}^\infty \left\{\frac{1}{LLn_K} \sup_{\theta\in S} \frac{U_\theta(\gamma_\theta(n_K))-U_\theta(\gamma_\theta(n_K)/dLLn_K)}{M_\theta(\gamma_\theta(n_K)/LLn_K)}\right\}^r\end{aligned} \tag{31}$$

since $r>\dfrac{3}{1-\delta}$ implies $r>3+r\delta>3-\delta+r\delta$. Thus (20), hence (17) holds, and the proof that $\overline{\lim_n} \|\Gamma_n S_n\|<\infty$ a.s. holds is complete in the symmetric case.

For the general case, we let $X', X_1', X_2', \ldots$, be i.i.d. copies of X, independent of $X, X_1, X_2, \ldots$, and we put $S_n'=X_1'+\ldots+X_n'$. We suppose that X satisfies (3); then

$$\sum_n P(\|\Gamma_n(X-X')\|>\varepsilon)\leqq 2\sum_n P(\|\Gamma_n X\|>\varepsilon/2)<\infty$$

(via Lemma 3), so that $X-X'$ satisfies (3) as well. By the symmetric result, $\overline{\lim_n} \|\Gamma_n(S_n-S_n')\|=M<\infty$ a.s., for some constant M. We claim that

$\overline{\lim_n} \|\Gamma_n S_n\| \leqq 2M$ a.s., and we accomplish the proof by showing for each $\beta > 0$

(i) $\overline{\lim_n} \|\Gamma_n(S_n - nEXI(\|\Gamma_n X\| \leqq \beta))\| \leqq 2M$ a.s. and

(ii) $\lim_{n\to\infty} nE\Gamma_n XI(\|\Gamma_n X\| > \beta) = 0$, recalling that $EX = 0$.

For (i), by Fubini's theorem we have vectors $\{c_n\}$ with $\overline{\lim_n} \|\Gamma_n(S_n - c_n)\| = M$ a.s. We claim that $\Gamma_n(S_n - nEXI(\|\Gamma_n X\| \leqq \beta)) \xrightarrow{P} 0$ for each $\beta > 0$. Assuming this for the moment, and given $\varepsilon > 0$, we have existence of sample points ω_n with $\|\Gamma_n(S_n(\omega_n) - c_n)\| \leqq M + \varepsilon$ (true for almost all sample points) and $\|\Gamma_n(S_n(\omega_n) - nEXI(\|\Gamma_n X\| \leqq \beta))\| \leqq \varepsilon$ (true for a set of sample points of large probability) for all sufficiently large n.

It follows that $\overline{\lim_n} \|\Gamma_n(c_n - nEXI(\|\Gamma_n X\| \leqq \beta))\| \leqq M + \varepsilon + \varepsilon = M + 2\varepsilon$, and thus that

$$\begin{aligned}\overline{\lim_n} \|\Gamma_n(S_n - nEXI(\|\Gamma_n X\| \leqq \beta)) &\leqq \overline{\lim_n} \|\Gamma_n(S_n - c_n)\| + \overline{\lim_n} \|\Gamma_n(c_n - nEXI(\|\Gamma_n X\| \leqq \beta))\| \\ &\leqq M + (M + 2\varepsilon) = 2M + 2\varepsilon, \quad \text{a.s.}\end{aligned}$$

Letting $\varepsilon \to 0$ gives (i), modulo the fact that $\Gamma_n(S_n - nEXI(\|\Gamma_n X\| \leqq \beta)) \xrightarrow{P} 0$.

We already know that $\Gamma_n(S_n - S_n') \xrightarrow{P} 0$ (because of Lemma 2(d) and $\mathscr{L}(T_n(S_n - S_n')) \xrightarrow{W} \gamma * \gamma$, where $*$ denotes convolution), so let $\beta > 0$, $\varepsilon > 0$, and recall that (3) implies, by Lemma 3(f) and an easy filling-in argument, that $\lim_n nP(\|\Gamma_n X\| > \lambda) = 0$ for each $\lambda > 0$. Thus,

$$\begin{aligned}P&\left(\left\|\Gamma_n \sum_{j=1}^n (X_j I(\|\Gamma_n X_j\| \leqq \beta) - X_j' I(\|\Gamma_n X_j'\| \leqq \beta))\right\| > \varepsilon\right) \\ &\leqq P(\|\Gamma_n(S_n - S_n')\| > \varepsilon) + P \text{ (for some } j \leqq n, \text{ either } \|\Gamma_n X_j\| > \beta \text{ or } \|\Gamma_n X_j'\| > \beta) \\ &\leqq P(\|\Gamma_n(S_n - S_n')\| > \varepsilon) + 2nP(\|\Gamma_n X\| > \beta) \to 0, \quad \text{as } n \to \infty. \qquad (32)\end{aligned}$$

Now (32) implies that $Y_n = \Gamma_n \sum_{j=1}^n (X_j I(\|\Gamma_n X_j\| \leqq \beta) - X_j' I(\|\Gamma_n X_j'\| \leqq \beta)) \xrightarrow{P} 0$. Thus $J\left(Y_n, \frac{1}{(8)(3^2)}\right) \to 0$ as $n \to \infty$, and an application of Lemma 6.1 in Kuelbs and Zinn (1983) with $r = 2$ to the r.v.'s $\Gamma_n(X_j I(\|\Gamma_n X_j\| \leqq \beta) - X_j' I(\|\Gamma_n X_j'\| \leqq \beta))$ (which are uniformly bounded by 2β in norm) gives $E\|Y_n\|^2 \leqq 18(4J(Y_n, 1/72) + (2\beta)^2) \to 72\beta^2$, as $n \to \infty$. Since $Y_n \xrightarrow{P} 0$, standard uniform integrability results (see, e.g., Chung (1968), Theorem 4.5.2) give $E\|Y_n\| \to 0$.

But Jensen's inequality and the independence of $\{X_j\}$, $\{X_j'\}$ show that

$$E\|Y_n\| \geqq E\left\|\Gamma_n\left(\sum_{j=1}^n (X_j I(\|\Gamma_n X_j\| \leqq \beta) - EX_j I(\|\Gamma_n X_j\| \leqq \beta))\right)\right\|,$$

and thus by Markov's inequality we see

$$\Gamma_n\left(\sum_{j=1}^n X_j I(\|\Gamma_n X_j\| \leqq \beta) - nEXI(\|\Gamma_n X\| \leqq \beta)\right) \xrightarrow{P} 0.$$

But then,

$$P(\|\Gamma_n(S_n - nEXI(\|\Gamma_n X\| \leqq \beta))\| > \varepsilon)$$
$$\leqq P\left(\left\|\Gamma_n\left(\sum_{j=1}^{n} X_j I(\|\Gamma_n X_j\| \leqq \beta) - nEXI(\|\Gamma_n X\| \leqq \beta)\right)\right\| > \varepsilon\right) + nP(\|\Gamma_n X\| > \beta) \to 0,$$

completing the proof of (i).

To prove (ii), we will show, in analogy with constant normalizations case, that $\lim_{n\to\infty} nE\|T_n X\| I(\|T_n X\| > \beta) = 0$. That this will suffice follows from

$$nE\|\Gamma_n X\| I(\|\Gamma_n X\| > \beta) = \frac{n}{LLn} E\|T_{[n/LLn]}X\| I(\|T_{[n/LLn]}X\| > \beta LLn)$$
$$\leqq \frac{n}{LLn} E\|T_{[n/LLn]}X\| I(\|T_{[n/LLn]}X\| > \beta) \to 0.$$

Now

$$nE\|T_n X\| I(\|T_n X\| > \beta) = n\int_0^\infty P(\|T_n X\| I(\|T_n X\| > \beta) > t)\,dt$$
$$= \beta nP(\|T_n X\| > \beta) + n\int_\beta^\infty P(\|T_n X\| > t)\,dt. \tag{33}$$

But $nP(\|T_n X\| > \beta) \to 0$ because of the Central Limit Theorem for Gaussian limits.

To estimate the last term in (33), let $\{\theta_i : 1 \leqq i \leqq d\}$ denote the PONB at time n, and observe that for n sufficiently large,

$$n\int_\beta^\infty P(\|T_n X\| > t)\,dt \leqq n\int_\beta^\infty \sum_{i=1}^{d} P\left(|\theta_i(X)| > \frac{t}{d} d_{\theta_i}(n)\right) dt$$
$$= n\sum_{i=1}^{d} \int_\beta^\infty P\left(|\theta_i(X)| > \frac{t}{d} d_{\theta_i}(n)\right) dt$$
$$\leqq dn \sup_{\theta\in S} \int_\beta^\infty P\left(|\theta(X)| > \frac{t}{d} d_\theta(n)\right) dt$$
$$= d^2 n \sup_{\theta\in S} \left(\int_{\beta d_\theta(n)/d}^\infty P(|\theta(X)| > u)\,du / d_\theta(n)\right)$$
$$= d^2 n \sup_{\theta\in S} \frac{1}{d_\theta(n)} \int_{\beta d_\theta(n)/d}^\infty \frac{u^2 P(|\theta(X)| > u)}{M_\theta(u)} \frac{M_\theta(u)}{u^2}\,du. \tag{34}$$

In (34) we use the Hahn-Klass Condition (1) and the fact that $d_\theta(n) \to \infty$ uniformly in $\theta \in S$ (indeed, for large n, $d_\theta(n) \geqq \sqrt{n}$) to see that we need only show

$$\limsup_{n\to\infty} \sup_{\theta\in S} \frac{n}{d_\theta(n)} \int_{\beta d_\theta(n)/d}^\infty \frac{M_\theta(u)}{u^2}\,du < \infty.$$

Now the "uniformly slowly varying" property of $M_\theta(\cdot)$, which derives from the Hahn-Klass Condition (1), can be used to adapt the proof of Lemma 8 (q.v.) to show that given $\delta>0$ and $C>0$, there exist n_0 and $D>0$ such that for all $n\geqq n_0$, all $j\geqq 1$ and all $\theta\in S$,

$$M_\theta(C2^j d_\theta(n))\leqq D2^{j\delta}M_\theta(d_\theta(n)).$$

Choose $\delta<1$; then for $n\geqq n_0$,

$$\frac{n}{d_\theta(n)}\int_{\beta d_\theta(n)/d}^{\infty}\frac{M_\theta(u)}{u^2}du=\frac{n}{d_\theta(n)}\sum_{j=1}^{\infty}\int_{2^{j-1}\beta d_\theta(n)/d}^{2^j\beta d_\theta(n)/d}M_\theta(u)/u^2du$$
$$\leqq\sum_{j=1}^{\infty}nM_\theta(2^j\beta d_\theta(n)/d)d/(\beta 2^j d_\theta^2(n)) \tag{35}$$

(using the monotonicity property of $M_\theta(\cdot)$)

$$\leqq\frac{Dd}{\beta}\sum_{j=1}^{\infty}2^{j\delta}nM_\theta(d_\theta(n))/(2^j d_\theta^2(n))$$

(uniformly in $\theta\in S$)

$$=\frac{Dd}{\beta}\sum_{j=1}^{\infty}(2^{1-\delta})^{-j}<\infty,$$

where in the last line of (35) we used the defining property (5) of d_θ, namely $d_\theta^2(n)=nM_\theta(d_\theta(n))$. Thus desymmetrization is complete, and $\overline{\lim_n}\|\Gamma_n S_n\|<\infty$ in the general case.

It remains to show that $\overline{\lim_n}\|\Gamma_n S_n\|>0$, and that the cluster set of $\{\Gamma_n S\}$ contains $\{x\in\mathbb{R}^d\colon \|x\|\leqq\sqrt{2}\}$. The technique depends on the following moderate deviations lemma patterned after de Acosta and Kuelbs (1983), Lemma 3.1. Because the clustering result does not depend on the refined regularity in Lemma (3), we remark that the proof we present can be extended easily to the Banach space case, under the additional assumption due to Urbanik (1978) that $\{T_n T_m^{-1}\colon m\leqq n\}$ is precompact, where $\mathscr{L}(T_n S_n)\xrightarrow{w}\gamma$ in the Banach space. Since the proof *does* assume boundedness of $\|\Gamma_n S_n\|$, however, we prefer to proceed only in the finite-dimensional case. In Proposition 5 the additional assumption $a_n^4/n^3\to 0$ is required for infinite dimensions, to compensate for the "unknown" fact there that $\Gamma_n S_n\xrightarrow{P}0$, as the reader may easily check.

Proposition 5. *Assume $EX=0$, $X\in\mathrm{GDA}(\gamma,\{T_n\})$. Let U be convex and open in $\mathbb{R}^d$. Suppose that $0<\frac{a_n}{n}\to 0$, but $\frac{a_n^2}{n}\nearrow\infty$. Let $b_n=[n^2/a_n^2]$. Then*

$$\liminf_{n\to\infty}\frac{n}{a_n^2}\log P\left(\frac{n}{a_n^2}T_{b_n}S_n\in U\right)\geqq\limsup_{t\to\infty}\frac{1}{t^2}\log\gamma(tU).$$

Proof. Let $\varepsilon>0$, $t>0$, put $U^\varepsilon=\{y\colon \inf_{u\in U^c}\|y-u\|>\varepsilon\}$. Then U^ε is open and convex. Write $p_n=\left[\frac{n^2t^2}{a_n^2}\right]$, $q_n=\left[\frac{n}{p_n}\right]$. (Note $|n-p_nq_n|=O(b_n)$ if $a_n^4/n^3\to 0$.)

Then, using independence and stationarity,

$$P\left(\frac{n^2}{a_n^2} T_{b_n} S_{p_n q_n} \in U^\varepsilon\right) = P\left(T_{b_n} S_{p_n q_n} \in \frac{a_n^2}{n} U^\varepsilon\right)$$
$$\geqq P\left(T_{b_n} S_{p_n} \in \frac{a_n^2}{n q_n} U^\varepsilon\right)^{q_n}. \tag{37}$$

Since $q_n \sim \frac{a_n^2}{nt^2}$ and $p_n/b_n \to t^2$, it is easy to see that $\mathscr{L}(T_{b_n} S_{p_n}) \xrightarrow{w} \gamma^{t^2}$, and $\gamma^{t^2}(E) = \gamma(tE)$ for Borel sets E. An easy way to see this is Donsker's Invariance Principle. Thus (37), after taking logarithms and letting $n \to \infty$, yields

$$\varliminf_n \frac{n}{a_n^2} \log P\left(\frac{n}{a_n^2} T_{b_n} S_{p_n q_n} \in U^\varepsilon\right) \geqq \varliminf_n \frac{n}{a_n^2} q_n \log P\left(T_{b_n} S_{p_n} \in \frac{a_n^2}{n q_n} U^\varepsilon\right)$$
$$= \frac{1}{t^2} \lim_{n\to\infty} \log P\left(T_{b_n} S_{p_n} \in \frac{a_n^2}{n q_n} U^\varepsilon\right)$$
$$= \frac{1}{t^2} \log \gamma(t U^\varepsilon). \tag{38}$$

Letting $\varepsilon \to 0$, $t \to \infty$ gives the result, after observing that independence gives

$$\log P\left(\frac{n}{a_n^2} T_{b_n} S_n \in U\right) \geqq \log P\left(\frac{n}{a_n^2} T_{b_n} S_{p_n q_n} \in U^\varepsilon\right) + \log P\left(\frac{n}{a_n^2} \|T_{b_n} S_{|n-p_n q_n|}\| < \varepsilon\right),$$

and that the latter term goes to zero using $n - p_n q_n = o(n)$, Lemma 2(d) (which states that $\left\|\frac{n}{a_n^2} T_{b_n} T_n^{-1}\right\| \leqq \frac{n}{a_n^2} \left(\frac{n}{b_n}\right)^{\frac{1}{2}+\delta}$, for any $0 < \delta < 1/2$ and all large n, $\to 0$ since $a_n^2/n \to \infty$), and again for simplicity Donsker's Invariance Principle.

Lemma 6. *Under the assumptions of Proposition 5,*

$$\liminf_{n\to\infty} \frac{n}{a_n^2} P\left(\left\|\frac{n}{a_n^2} T_{b_n} S_n - b\right\| < \varepsilon\right) \geqq -\frac{\|b\|^2}{2},$$

for every $\varepsilon > 0$.

Proof. The Cameron-Martin formula allows evaluation of $\varliminf_t \frac{1}{t^2} \log \gamma(tU)$, where $U = \{y \in \mathbb{R}^d : \|y - b\| < \varepsilon\}$. For proof see de Acosta and Kuelbs (1983), Lemma 3.2, noting that the "Reproducing Kernel Hilbert Space" induced by γ is just $(\mathbb{R}^d, \|\cdot\|)$, since γ is standardized normal, and $\|\cdot\|$ is Euclidean.

Now to show that $\|b\| \leqq \sqrt{2}$ implies b is in the cluster set of $\{\Gamma_n S_n\}$ a.s., it suffices to assume $\|b\| < \sqrt{2}$ (since the cluster set is closed). It is easy to see that the proof of Proposition 5 applies to $\frac{n}{a_n^2} T_{b_n} S_{d_n}$, as long as $d_n - n = O(b_n)$. So let $n_K = K^K$, and note that $n_K - n_{K-1} = O(b_{n_K})$. Thus, from Lemma 8 we have, given $\delta > 0$ such that $\frac{\|b\|^2}{2} + \delta < 1$ and putting $a_n^2 = nLLn$, $\varepsilon > 0$, that for K_0 sufficiently large,

$$\sum_{K \geqq K_0} P(\|\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}}) - b\| < \varepsilon)$$
$$= \sum_{K \geqq K_0} P(\|\Gamma_{n_K} S_{n_K - n_{K-1}} - b\| < \varepsilon) \geqq \sum_K \exp\left\{-\left(\frac{\|b\|^2}{2} + \delta\right) LLn_K\right\}$$
$$= \sum_{K \geqq K_0} (K \log K)^{-\left(\frac{\|b\|^2}{2} + \delta\right)} = \infty. \tag{39}$$

Thus the Borel-Cantelli lemma applies to the independent sequence $\{\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}}): K \geqq 1\}$ to show that b is almost surely a cluster point. But Lemma 2(d) shows that $\|\Gamma_{n_K}\Gamma_{n_{K-1}}^{-1}\| \to 0$, and together with $\overline{\lim}_{n\to\infty} \|\Gamma_n S_n\| < \infty$ a.s., we use $\|\Gamma_{n_K} S_{n_K} - b\| \leqq \|\Gamma_{n_K}(S_{n_K} - S_{n_{K-1}}) - b\| + \|\Gamma_{n_K}\Gamma_{n_{K-1}}^{-1}\| \, \|\Gamma_{n_{K-1}} S_{n_{K-1}}\|$ to complete the proof of Theorem 1.

3. Technical Lemmas and Their Proofs

In this section we state and prove technical results used (with proof deferred) in the proof of Theorem 1.

Lemma 7. *Given* $\{x_j\} \subset \mathbb{R}^d$ *with* $\overline{\lim}_{j\to\infty} \|\Gamma_{n_j} x_j\| = C < \infty$, *we have*

$$\overline{\lim}_{K\to\infty} \left\|\Gamma_{n_K} \sum_{j=1}^K x_j\right\| \leqq C\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right).$$

Proof. We use

$$\left\|\Gamma_{n_K} \sum_{j=1}^K x_j\right\| = \left\|\sum_{j=1}^K \Gamma_{n_K}\Gamma_{n_j}^{-1}(\Gamma_{n_j} x_j)\right\| \leqq \left(\sum_{j=N}^K \|\Gamma_{n_K}\Gamma_{n_j}^{-1}\|\right)(C+\varepsilon) + \left\|\Gamma_{n_K} \sum_{j=1}^{N-1} x_j\right\|,$$

where $\varepsilon > 0$, and N is so larg that $j \geqq N$ implies $\|\Gamma_{n_j} x_j\| < C + \varepsilon$. We claim $\overline{\lim}_{K\to\infty} \sum_{j=1}^K \|\Gamma_{n_K}\Gamma_{n_j}^{-1}\| \leqq \frac{\sqrt{2}}{\sqrt{2}-1}$, which, since $\|\Gamma_{n_K}\| \to 0$ as $K \to \infty$, will complete the proof.

Choose M so large that (by Lemma 2(b)) $j \geqq M$ implies $\|\Gamma_{n_{j+1}}\Gamma_{n_j}^{-1}\| < 1/(\sqrt{2} - \varepsilon)$. As $\Gamma_{n_K}\Gamma_{n_j}^{-1} = \prod_{i=j}^{K-1} \Gamma_{n_{i+1}}\Gamma_{n_i}^{-1}$, we have $\|\Gamma_{n_K}\Gamma_{n_j}^{-1}\| \leqq \left(\frac{1}{\sqrt{2}-\varepsilon}\right)^{k-j}$ when $K \geqq j \geqq M$.

But then $\sum_{j=1}^K \|\Gamma_{n_K}\Gamma_{n_j}^{-1}\| = \sum_{j=1}^{M-1} \|\Gamma_{n_K}\Gamma_{n_j}^{-1}\| + \sum_{j=M}^K \|\Gamma_{n_K}\Gamma_{n_j}^{-1}\|$, when $K \geqq M$, and it is easy to see that $\|\Gamma_{n_K}\| \max_{j \leqq M-1} \|\Gamma_{n_j}^{-1}\| \to 0$ as $K \to \infty$, since $\|\Gamma_{n_K}\| \to 0$. Yet,

$$\sum_{j=M}^K \|\Gamma_{n_K}\Gamma_{n_j}^{-1}\| \leqq \sum_{j=M}^K \left(\frac{1}{\sqrt{2}-\varepsilon}\right)^{K-j} = \sum_{i=0}^{K-M} \left(\frac{1}{\sqrt{2}-\varepsilon}\right)^i \leqq \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{2}-\varepsilon}\right)^i$$
$$= 1/(1 - (\sqrt{2} - \varepsilon)^{-1})$$

(geometric series); letting $\varepsilon \to 0$ gives the bound $\sqrt{2}/(\sqrt{2} - 1)$.

Proof of Lemma 3. (a) implies (b): Assume $\sum_n P(\|\Gamma_n X\| > K_0)$ diverges for some K_0. Clearly the series diverges for all $K \leqq K_0$, so fix $K > K_0$. Fix N so large that $\frac{2}{\sqrt{N}} < \frac{K_0}{K}$. Now Lemma 2(b) shows $\lim_{n\to\infty} \max_{0\leqq j\leqq N-1} \|\Gamma_{Nn+j}\Gamma_n^{-1}\| = \frac{1}{\sqrt{N}}$ $\left(\text{since } \frac{n}{Nn+j} \to \frac{1}{N},\ 0 \leqq j \leqq N-1\right)$. So choose n_0 so large that $n > n_0$ implies $\|\Gamma_{Nn+j}\Gamma_n^{-1}\| < \frac{2}{\sqrt{N}}$ when $0 \leqq j \leqq N-1$. Then, $n \geqq n_0$ implies

$$\begin{aligned}
\max_{0\leqq j\leqq N-1} P(\|\Gamma_{Nn+j}X\| > K_0) &\leqq P(\max_{0\leqq j\leqq N-1} \|\Gamma_{Nn+j}X\| > K_0) \\
&\leqq P((\max_{0\leqq j\leqq N-1} \|\Gamma_{Nn+j}\Gamma_n^{-1}\|)\|\Gamma_n X\| > K_0) \\
&\leqq P\left(\frac{2}{\sqrt{N}}\|\Gamma_n X\| > K_0\right) = P\left(\|\Gamma_n X\| > \frac{K_0\sqrt{N}}{2}\right) \\
&\leqq P(\|\Gamma_n X\| > K), \quad \text{by choice of } N.
\end{aligned}$$

Thus,

$$\infty = \sum_{m=Nn_0}^{\infty} P(\|\Gamma_m X\| > K_0) = \sum_{n=n_0}^{\infty} \sum_{j=0}^{N-1} P\left(\|\Gamma_{Nn+j}X\| > K_0\right) \leqq \sum_{n=n_0}^{\infty} NP(\|\Gamma_n X\| > K).$$

Thus, divergence for one K_0 implies divergence for every K.

Now (a) iff (d) iff (e) follows from the Borel-Cantelli lemma and the fact that (a) implies (b), since $\{X_n\}$ are independent. For (b) iff (f), we use Lemma 2(c) and proceed in manner similar to that above, in (a) implies (b), with $n_K - n_{K-1} = n_{K-1}$ playing the role of N.

Finally, for (b) iff (c), we use the result of Hahn and Klass that if $\{\theta_i : i = 1, \ldots, d\} = \{\theta_i(n) : i = 1, \ldots, d\}$ represents the PONB at time n, then $d_\theta^2(n) \approx \sum_{i=1}^{d} (\theta, \theta_i)^2 d_{\theta_i}(n)^2$ (recall $\approx$ means asymptotic uniformly in $\theta \in S$).

Thus, given $c > 1$, there exists n_0 so that uniformly in $\theta \in S$ and for all $n \geqq n_0$, we have (letting $\{\theta_i = \theta_i(n/LLn) : i = 1, \ldots, d\}$ now denote the PONB at stage n/LLn)

$$\sum_{i=1}^{d} (\theta, \theta_i)^2 \gamma_{\theta_i}^2(n) \leqq c\gamma_\theta^2(n). \tag{40}$$

Now, suppose that for $i = 1, 2, \ldots, d$ we had $|\theta_i(X)| \leqq \gamma_{\theta_i}(n)/d\sqrt{c}$. Then we would also have

$$\begin{aligned}
|\theta(X)| &= \left|\sum_{i=1}^{d} (\theta, \theta_i)\theta_i(X)\right| \\
&\leqq \sum_{i=1}^{d} |(\theta, \theta_i)\theta_i(X)| \\
&\leqq \sum_{i=1}^{d} |(\theta, \theta_i)|\gamma_{\theta_i}(n)/d\sqrt{c} \\
&\leqq \sum_{i=1}^{d} \left(\sum_{h=1}^{d} (\theta, \theta_j)^2 \gamma_{\theta_j}^2(n)\right)^{1/2} \Big/ d\sqrt{c} \\
&\leqq \sum_{i=1}^{d} (\sqrt{c}\,\gamma_\theta(n))/d\sqrt{c} = \gamma_\theta(n), \quad \text{by (40)}.
\end{aligned}$$

Hence $|\theta(X)|>\gamma_\theta(n)$ implies for some $i=1,2,\ldots,d$, we have $|\theta_i(X)|>\gamma_{\theta_i}(n)/d\sqrt{c}$, and thus

$$\begin{aligned}P(|\theta(X)|>\gamma_\theta(n)) &\leqq \sum_{i=1}^{d} P(|\theta_i(X)|>\gamma_{\theta_i}(n)/d\sqrt{c})\\ &\leqq \sum_{i=1}^{d} P\left(\sum_{j=1}^{d}\frac{\theta_j^2(X)}{\gamma_{\theta_j}^2(n)}>\frac{1}{d^2c}\right)\\ &=dP(\|\tilde{\Gamma}_nX\|>1/d\sqrt{c})\\ &\leqq dP(\|\Gamma_nX\|>2/d\sqrt{c}),\end{aligned}$$

using $\tilde{\Gamma}_n\Gamma_n^{-1}\to I$ by (21). Thus, taking suprema above gives

$$\sup_{\theta\in S} P(|\theta(X)|>\gamma(n,\theta))\leqq dP(\|\Gamma_nX\|>2/d\sqrt{c}).$$

Thus (3) and (a) implies (b), above, imply (c). On the other hand,

$$\begin{aligned}P(\|\Gamma_nX\|>\varepsilon)\leqq P(\|\tilde{\Gamma}_nX\|>\varepsilon/2)=P\left(\sum_{i=1}^{d}\frac{\theta_i^2(X)}{\gamma_{\theta_i}^2(n)}>\frac{\varepsilon^2}{4}\right)\leqq \sum_{i=1}^{d} P\left(|\theta_i(X)|>\frac{\varepsilon}{2\sqrt{d}}\gamma_{\theta_i}(n)\right)\\ \leqq d\sup_{\theta\in S} P\left(|\theta(X)|>\frac{\varepsilon}{2\sqrt{d}}\gamma_\theta(n)\right).\end{aligned}$$

Thus the integral test, monotonicity of $\gamma(\theta,\cdot)$ and (41) show (c) implies (b), and the proof of Lemma 3 is complete.

Lemma 8. *For each* $\delta>0$,

$$\lim_{t\to\infty}\sup_{\theta\in S}\frac{U_\theta(\gamma_\theta(t))}{(LLt)^\delta U_\theta(\gamma_\theta(t)/LLt)}=0.$$

Proof. The Equivalence Lemma of Hahn and Klass (1980a) implies that

$$\lim_{t\to\infty}\sup_{\theta\in S}\frac{U_\theta(2t)}{U_\theta(t)}=1,$$

i.e., that the functions $U_\theta(\cdot)$ are uniformly slowly varying for $\theta\in S$. Let log denote logarithm to the base two, and put $a(t)=[\log(LLt)]$. Let $\varepsilon>0$. We have, since $1\leqq LLt/2^{a(t)}\leqq 2$,

$$\begin{aligned}b(t)\equiv\sup_{\theta\in S}\frac{U_\theta(\gamma_\theta(t))}{U_\theta(\gamma_\theta(t)/LLt)}&\leqq 2\prod_{j=0}^{a(t)-1}\sup_{\theta\in S}\frac{U_\theta(\gamma_\theta(t)2^{j+1}/LLt)}{U_\theta(\gamma_\theta(t)2^j/LLt)}\\ &\leqq 2(1+\varepsilon)^{a(t)},\end{aligned}$$

whenever t is large enough that $\sup_{\theta\in S}\frac{U_\theta(2\sqrt{s})}{U_\theta(\sqrt{s})}<1+\varepsilon$ for all $s>t$, upon noting that uniformly in $\theta\in S$, for all sufficiently large s, we have $\gamma_\theta(s)\geqq\sqrt{s}$.

We thus have $\overline{\lim_{t\to\infty}}\frac{\log b(t)}{a(t)}=0$, after letting $\varepsilon\to 0$. It follows $\lim_{t\to\infty}\frac{b(t)}{(LLt)^\delta}=0$ for every $\delta>0$, as asserted.

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