EDGEWORTH EXPANSION

Consider the central limit theorem: For $X_1, \dots, X_n$ iid with mean μ and variance $\sigma^2 < \infty$,

$$\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \to N(0,1) \quad \text{in dist. as} \quad n \to \infty.$$

We might wish to know

- How fast is the convergence to Normality?
- What features of the distribution of $X_1, \dots, X_n$ affect the rate of convergence and how?

Edgeworth expansions help us to answer these questions. Moreover, they can be used to show that the boot works — and that it can provide a better approximation to the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ than the $N(0,1)$ distribution.

Theorem (1st and 2nd order Edgeworth Expansions):

Let $Y_1, \dots, Y_n$ be iid with $\mathbb{E}\, Y_1 = \mu$, $\operatorname{Var} Y_1 = \sigma^2 \in (0, \infty)$, $\mathbb{E}|Y_1|^3 < \infty$, $\mathbb{E}|Y_1|^4 < \infty$, and suppose

$$\left|\mathbb{E} \exp(\imath t Y_1)\right| < 1 \quad \text{for all} \quad t \neq 0. \quad \text{(Non-lattice)}$$

Then

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma} \le x\right) - \Psi_{n,3}(x) \right| = o(n^{-1/2})$$

as $n \to \infty$ and

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma} \le x\right) - \Psi_{n,4}(x) \right| = o(n^{-1}),$$

where

$$\Psi_{n,3}(x) = \Phi(x) - \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3}(x^2-1)\,\phi(x)$$

$$\Psi_{n,4}(x) = \Phi(x) - \left\{ \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3}(x^2-1) + \frac{1}{24\,n}\cdot\left(\frac{\mu_4}{\sigma^4}-3\right)\left(x^3-3x\right) + \frac{1}{72n}\cdot\frac{\mu_3^2}{\sigma^6}\left(x^5-10x^3+15x\right)\right\}\phi(x).$$

We now spend some time deriving the E.E.: Here I acknowledge my indebtedness to Dr. David Hunter at Penn State, whose notes I have essentially followed.

We will need to introduce Hermite polynomials:

__Hermite polynomials:__ The Hermite polynomials $H_1, H_2, \ldots$ are defined by the relation

$$\underbrace{(-1)^k}_{\text{b/c } \frac{\partial}{\partial x}\phi(x) \text{ always brings a minus sign down to the front.}} \frac{d^k}{dx^k}\phi(x) = H_k(x)\,\phi(x), \qquad \text{for } k=1,2,\ldots$$

We obtain, for $k=1,2,3$, the following:

$$\boxed{\frac{d}{dx}\left[H_k(x)\,\phi(x)\right] = (-1)^k \frac{d^{k+1}}{dx^{k+1}}\phi(x) = (-1)\,H_{k+1}(x)\,\phi(x)}$$

__k=1:__

$$\frac{d}{dx}\phi(x) = \frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$= -x\,\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$= (-1)^1 \underbrace{x}_{H_1(x).}\,\phi(x)$$

so $H_1(x) = x$ since $(-1)^1\phi^{(1)}(x) = H_1(x)\,\phi(x)$

__k=2:__

$$\frac{d^2}{dx^2}\phi(x) = \frac{d}{dx}\left[-x\,\phi(x)\right]$$

$$= -x\,\phi'(x) - \phi(x)$$

$$= x^2 \phi(x) - \phi(x)$$

$$= (x^2 - 1)\,\phi(x).$$

So $H_2(x) = x^2 - 1$, since $(-1)^2 \frac{d^2}{dx^2}\phi(x) = (x^2-1)\,\phi(x)$.

<u>**k=3:**</u> $\frac{d^3}{dx^3}\phi(x) = \frac{d}{dx}(x^2-1)\,\phi(x)$

$$= (x^2-1)\,\phi'(x) + 2x\,\phi(x)$$

$$= (x^2-1)(-1)x\,\phi(x)$$

$$= (-1)\left[x^3 - x - 2x\right]\phi(x)$$

$$= (-1)\underbrace{\left[x^3 - 3x\right]}_{H_3(x)}\phi(x).$$

So $H_3(x) = x^3 - 3x$, since $(-1)^3 \frac{d^3}{dx^3} = (x^3-3x)\,\phi(x)$.

Doing more work, we can obtain

$$H_4(x) = x^4 - 6x^2 + 3$$

$$H_5(x) = x^5 - 10x^3 + 15x.$$

Note that the Hermite polynomials appear in the Edgeworth Expansions!

Next, we will use the fact that if X is a rv with characteristic function ψ_X satisfying $\int_{-\infty}^{\infty} |\psi_X(t)|\, dt < \infty$, then the density of X is given by

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\imath t x)\, \psi_X(t)\, dt, \quad \text{for all } x \in \mathbb{R}$$

(This is called the inversion formula)

to obtain the following useful identity:

Useful Identity:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\imath t x}\, e^{-t^2/2}\, (\imath t)^k \, dt = \frac{(-1)^k}{2\pi} \int_{-\infty}^{\infty} \frac{d^k}{dx^k}\, e^{-\imath t x}\, e^{-t^2/2}\, dt$$

$$= (-1)^k \frac{d^k}{dx^k} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\imath t x}\, e^{-t^2/2}\, dt}$$

$$= (-1)^k \frac{d^k}{dx^k} \phi(x)$$

$= \phi(x)$ by the inversion formula, since $\exp(-t^2/2)$ is the characteristic function of the $N(0,1)$ dist.

$$= H_k(x)\, \phi(x).$$

Derivation of Edgeworth expansions:

We will simplify slightly by assuming, without loss of generality, that $X_1, \dots, X_n$ are iid with

$$\mathbb{E} X_1 = 0 \qquad \mathbb{E} X_1^3 = \gamma$$

$$\mathbb{E} X_1^2 = 1 \qquad \mathbb{E} X_1^4 = \tau < \infty$$

and that

$$\left| \mathbb{E} \exp(\imath t X_1) \right| < 1 \qquad \text{for all } t \neq 0.$$

We study the distribution of the standardized sum

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i = \frac{\sqrt{n}\,(\bar{X}_n - \overset{\mu}{0})}{\underset{\sigma}{1}} .$$

Begin by constructing the characteristic function of S_n:

$$\begin{aligned}
\psi_{S_n}(t) &= \mathbb{E} \exp\left(\iota t S_n\right) \\
&= \mathbb{E} \exp\left(\frac{\iota t}{\sqrt{n}} \sum_{i=1}^{n} X_i \right) \\
&= \mathbb{E} \prod_{i=1}^{n} \exp\left(\frac{\iota t}{\sqrt{n}} X_i \right) \\
&\overset{(\text{ind})}{=} \prod_{i=1}^{n} \mathbb{E} \exp\left(\frac{\iota t}{\sqrt{n}} X_i \right) \\
&= \left[\mathbb{E} \exp\left(\frac{\iota t}{\sqrt{n}} X_1 \right) \right]^n
\end{aligned}$$

The non-lattice condition comes in here. We need $\mathbb{E}\exp(\iota t X/\sqrt{n}) < 1$ so that we can raise it to the power n without making it diverge.

Now make a Taylor expansion of $\mathbb{E}\exp\left(\frac{\iota t}{\sqrt{n}} X\right)$ around $t = 0$:

$$\begin{aligned}
\mathbb{E}\exp\left[\frac{\iota t}{\sqrt{n}} X \right] &= \mathbb{E}\Bigg(\Bigg[1 + \frac{\iota X}{\sqrt{n}} \cdot t + \frac{1}{2}\left(\frac{\iota X}{\sqrt{n}}\right)^2 t^2 + \frac{1}{6}\left(\frac{\iota X}{\sqrt{n}}\right)^3 t^3 \\
&\qquad + \frac{1}{24}\left(\frac{\iota X}{\sqrt{n}}\right)^4 t^4 \Bigg] + o(n^{-2}) \Bigg) \\
&= \left(1 + \frac{t^2}{2n} \right) + \frac{\gamma}{6}\,\frac{(\iota t)^3}{n^{3/2}} + \frac{\tau}{24}\,\frac{(\iota t)^4}{n^2} + o(n^{-2})
\end{aligned}$$

$$\left[e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \right]$$

Now we want to raise this to the n^{th} power:

$$\left(\mathbb{E}\ \exp\left[\frac{2t}{\sqrt{n}}X\right]\right)^n = \left[\left(1 - \frac{t^2}{2n}\right) + \frac{\gamma}{6}\frac{(2t)^3}{n^{3/2}} + \frac{\tau}{24}\frac{(2t)^4}{n^2} + o(n^{-2})\right]^n$$

$$\frac{n!}{(n-1)!\,1} = n$$

$$= \left(1 - \frac{t^2}{2n}\right)^n + n\left(1 - \frac{t^2}{2n}\right)^{n-1}\frac{\gamma}{6}\frac{(2t)^3}{n^{3/2}}$$

$$+ n\left(1 - \frac{t^2}{2n}\right)^{n-1}\frac{\tau}{24}\frac{(2t)^4}{n^2}$$

$$+ \underbrace{\frac{n(n-1)}{2}}_{\frac{n!}{(n-2)!\,2!}}\left(1 - \frac{t^2}{2n}\right)^{n-2}\left[\frac{\gamma}{6}\frac{(2t)^3}{n^{3/2}}\right]^2 + o(n^{-1})$$

$$= \left(1 - \frac{t^2}{2n}\right)^n + \left(1 - \frac{t^2}{2n}\right)^{n-1}\left[\frac{\gamma}{6}\frac{(2t)^3}{n^{1/2}} + \frac{\tau}{24}\frac{(2t)^4}{n}\right]$$

$$+ \left(1 - \frac{t^2}{2n}\right)^{n-2}\frac{(n-1)}{n^2}\frac{\gamma^2(2t)^6}{72} + o(n^{-1}),$$

where the second equality comes from the Multinomial Theorem

$$(a_1 + \dots + a_m)^n = \sum_{\substack{n_1,\dots,n_m \in \{0,\dots,n\},\\ n_1+\dots+n_m = n}} \left(\frac{n!}{n_1! \cdot \ldots \cdot n_m!}\right) a_1^{n_1} \cdot \ldots \cdot a_m^{n_m}.$$

Next, we use the fact that for each nonnegative integer k,

$$(\star)\qquad \left(1+\frac{a}{n}\right)^{n-k} = e^{a}\left(1-\frac{a(a+k)}{2n}\right) + o\left(\frac{1}{n}\right) \quad \text{as } n\to\infty$$

to write

$$\left(1-\frac{t^2}{2n}\right)^{n} = e^{-t^2/2}\left(1+\frac{t^2/2\,(-t^2/2)}{2n}\right) + o\left(\frac{1}{n}\right)$$

$$\left(1-\frac{t^2}{2n}\right)^{n-1} = e^{-t^2/2}\left(1+\frac{t^2/2\,(1-t^2/2)}{2n}\right) + o\left(\frac{1}{n}\right)$$

$$\left(1-\frac{t^2}{2n}\right)^{n-2} = e^{-t^2/2}\left(1+\frac{t^2/2\,(2-t^2/2)}{2n}\right) + o\left(\frac{1}{n}\right).$$

Plugging these into our expression for $\left(\mathbb{E}\exp\left(\frac{2t}{\sqrt{n}}X\right)\right)^n$ gives

$$\left(\mathbb{E}\exp\left[\frac{2t}{\sqrt{n}}X\right]\right)^n =$$

$$e^{-t^2/2}\Bigg[\left(1-\frac{t^4/4}{2n}\right) + \left(1-\frac{t^2/2\,(1-t^2/2)}{2n}\right)\left[\frac{\gamma}{6}\frac{(2t)^3}{n^{1/2}} + \frac{\tau}{24}\frac{(2t)^4}{n}\right]$$

$$+ \left(1-\frac{t^2/2\,(2-t^2/2)}{2n}\right)\underbrace{\frac{(n-1)}{n^2}}_{\text{throw out } \frac{1}{n^2}\frac{\gamma^3(2t)^6}{72}}\frac{\gamma^2(2t)^6}{72}\Bigg] + o(n^{-1})$$

$$= e^{-t^2/2}\left[1-\frac{t^4}{8n} + \frac{\gamma}{6}\frac{(2t)^3}{n^{1/2}} + \frac{\tau}{24}\frac{(2t)^4}{n} + \frac{1}{n}\frac{\gamma^2(2t)^6}{72}\right] + o(n^{-1})$$

$$= e^{-t^2/2}\left[1 + \frac{\gamma}{6}\frac{(\imath t)^3}{\sqrt{n}} + \frac{(\tau-3)}{24n}(\imath t)^4 + \frac{\gamma^2(\imath t)^6}{72\,n}\right] + o\left(\tfrac{1}{n}\right).$$

Remember, this is a Taylor expansion of ψ_{S_n}. Let

$$\tilde{\psi}_{S_n}(t) = e^{-t^2/2}\left[1 + \frac{\gamma}{6}\frac{(\imath t)^3}{\sqrt{n}} + \frac{(\tau-3)}{24n}(\imath t)^4 + \frac{\gamma^2(\imath t)^6}{72\,n}\right]$$

Now, to get an approximation to the cdf of S_n, we

(i) invert $\tilde{\psi}_{S_n}$ to get the corresponding pdf $\tilde{f}_{S_n}$

(ii) take the antiderivative of $\tilde{f}_{S_n}$.

Inversion of $\tilde{\psi}_{S_n}(t)$:

$$\tilde{f}_{S_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\,\tilde{\psi}_{S_n}(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\, e^{-t^2/2}\left[1 + \frac{\gamma}{6}\frac{(\imath t)^3}{\sqrt{n}} + \frac{(\tau-3)}{24n}(\imath t)^4 + \frac{\gamma^2(\imath t)^6}{72\,n}\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\, e^{-t^2/2}\,dt + \frac{\gamma}{6\sqrt{n}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\, e^{-t^2/2}(\imath t)^3\,dt$$

$$+ \frac{(\tau-3)}{24n}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\, e^{-t^2/2}(\imath t)^4\,dt$$

$$+ \frac{\gamma^2}{72\,n}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\, e^{-t^2/2}(\imath t)^6\,dt$$

$$= \phi(x) + \frac{\gamma}{6\sqrt{n}} H_3(x)\,\phi(x) + \frac{\tau-3}{24\,n} H_4(x)\,\phi(x)$$

$$+ \frac{\gamma^2}{72n} H_6(x)\,\phi(x).$$

So we have

$$\tilde{f}_{S_n}(x) = \phi(x)\left[1 + \frac{\gamma}{6\sqrt{n}} H_3(x) + \frac{(\tau-3)}{24n} H_4(x) + \frac{\gamma^2}{72n} H_6(x)\right].$$

Now take the antiderivative of $\tilde{f}_{S_n}(x)$, making use of the fact

$$\frac{d}{dx}\left[H_k(x)\,\phi(x)\right] = -H_{k+1}(x)\,\phi(x).$$

$$\begin{aligned}
\tilde{F}_{S_n}(x) &= \Phi(x) - \frac{\gamma}{6\sqrt{n}} H_2(x)\,\phi(x) - \frac{(\tau-3)}{24n} H_3(x)\,\phi(x) - \frac{\gamma^2}{72n} H_5(x)\,\phi(x) \\
&= \Phi(x) - \phi(x)\left[\frac{\gamma}{6\sqrt{n}} H_2(x) + \frac{(\tau-3)}{24n} H_3(x) + \frac{\gamma^2}{72n} H_5(x)\right] \\
&= \Phi(x) - \phi(x)\left[\frac{\gamma}{6\sqrt{n}}(x^2-1) + \frac{(\tau-3)}{24n}(x^3-3x) + \frac{\gamma^2}{72n}\overbrace{(x^5-10x^3+15x)}^{H_5(x)}\right].
\end{aligned}$$

This completes the derivation of the Edgeworth Expansions.

Note that

$$\gamma = \mathbb{E}X_1^3 = \mathbb{E}\left(\frac{X_1-0}{1}\right)^3 = \frac{\mu_3}{\sigma^3},$$

where $\mu_3 = \mathbb{E}(X_1-\mathbb{E}X_1)^3$ and $\sigma^2 = \mathbb{E}(X_1-\mathbb{E}X_1)^2$, and

$$\tau - 3 = \mathbb{E}\left(\frac{X_1-0}{1}\right)^4 = \frac{\mu_4}{\sigma^4} - 3,$$

where $\mu_4 = \mathbb{E}(X_1-\mathbb{E}X_1)^4$.

We can also find Edgeworth expansions for the studentized pivot

$$\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\hat{\sigma}_n}.$$

We state these results without derivation (see Hall (1992) for details).

Theorem (1st and 2nd order Edgeworth Expansions for studentized pivot):
Let $Y_1, \dots, Y_n$ be iid with $\mathbb{E}\,Y = \mu$, $\mathrm{Var}\,Y = \sigma^2 \in (0,\infty)$, $\mathbb{E}|Y|^3 < \infty$, $\mathbb{E}|Y|^4 < \infty$, and suppose

$$\left|\mathbb{E}\exp(\iota t Y)\right| < 1 \quad \text{for all} \quad t \neq 0. \quad \text{(Non-lattice)}$$

Then

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\,(\bar{Y}_n - \mu)}{\hat{\sigma}_n} \le x\right) - \tilde{\Psi}_{n,3}(x) \right| = o(n^{-1/2})$$

as $n \to \infty$ and

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\,(\bar{Y}_n - \mu)}{\hat{\sigma}_n} \le x\right) - \tilde{\Psi}_{n,4}(x) \right| = o(n^{-1}),$$

where

$$\tilde{\Psi}_{n,3}(x) = \Phi(x) + \frac{1}{6\sqrt{n}}\frac{\mu_3}{\sigma^3}(2x^2+1)\,\phi(x)$$

$$\tilde{\Psi}_{n,4}(x) = \Phi(x) + \left\{ \frac{1}{6\sqrt{n}}\frac{\mu_3}{\sigma^3}(2x^2+1) + \frac{1}{12n}\cdot\left(\frac{\mu_4}{\sigma^4} - 3\right)(x^3 - 3x) - \frac{1}{18n}\cdot\frac{\mu_3^2}{\sigma^6}\left(x^5 + 2x^3 - 3x\right) - \frac{1}{4n}(x^3+3x) \right\} \phi(x).$$

EDGEWORTH EXPANSION FOR THE BOOTSTRAP

Consider again the three pivots from the previous lecture

$$\sqrt{n}(\bar{X}_n-\mu), \qquad \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}, \qquad \text{and} \qquad \frac{\sqrt{n}(\bar{X}_n-\mu)}{\hat{\sigma}_n},$$

where $X_1,\dots,X_n$ are iid with $\mathbb{E}X_1=\mu$, $\mathrm{Var}\,X_1=\sigma^2$, $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X}_n)^2$ and the corresponding cdfs

$$G_{n,U}(x) = P\left(\sqrt{n}(\bar{X}_n-\mu) \le x\right)$$

$$G_n(x) = P\left(\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} \le x\right)$$

$$G_{n,S}(x) = P\left(\frac{\sqrt{n}(\bar{X}_n-\mu)}{\hat{\sigma}_n} \le x\right) \qquad \text{for all } x\in\mathbb{R}.$$

The Edgeworth expansion results, applied to these pivots, give

$$\sup_{x\in\mathbb{R}} \left| G_{n,U}(x) - \Phi(x/\sigma)\right| = O(n^{-1/2})$$

$$\sup_{x\in\mathbb{R}} \left| G_n(x) - \Phi(x)\right| = O(n^{-1/2})$$

$$\sup_{x\in\mathbb{R}} \left| G_{n,S}(x) - \Phi(x)\right| = O(n^{-1/2})$$

as $n\to\infty$.

We interpret these results by saying that the "accuracy" of the Normal approximation to $G_{n,U}$, G_n, and $G_{n,S}$ is of the order $O(n^{-1/2})$.

Now we present a result for the accuracy of the bootstrap estimators of the cdfs $G_{n,U}$, G_n, and $G_{n,S}$.

Result: For $X_1, \ldots, X_n$ iid with mean μ, variance $\sigma^2 \in (0,\infty)$, $\mathbb{E}X_1^3 = \mu_3$ and $\mathbb{E}X_1^4 = \mu_4 < \infty$, provided

$$\left| \mathbb{E} \exp(-\imath t X_1) \right| < 1 \quad \text{for all } t \neq 0,$$

we have

$$\text{(i)} \quad \sup_{x \in \mathbb{R}} \left| \hat{G}_{n,U}(x) - G_{n,U}(x) \right| = O_p(n^{-1/2})$$

$$\text{(ii)} \quad \sup_{x \in \mathbb{R}} \left| \hat{G}_n(x) - G_n(x) \right| = o_p(n^{-1/2})$$

$$\text{(iii)} \quad \sup_{x \in \mathbb{R}} \left| \hat{G}_{n,S}(x) - G_{n,S}(x) \right| = o_p(n^{-1/2})$$

as $n \to \infty$.

Note, the accuracy $o_p(n^{-1/2})$ is better than $O_p(n^{-1/2})$ and $O(n^{-1/2})$.

We refer to the property of this improved accuracy as **second-order correctness.**

The studentized bootstrap will, by the above result, perform better than the unstandardized bootstrap.

Sketch of Proof for Result:

For (i), we have, by the Edgeworth expansion result

$$\sup_{x \in \mathbb{R}} \left| G_{n,U}(x) - \left\{ \Phi(x/\sigma) - \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3} \left(\left(\frac{x}{\sigma}\right)^2 - 1 \right) \phi\left(\frac{x}{\sigma}\right) \right| = O(n^{-1})$$

and it can be shown that

$$\sup_{x \in \mathbb{R}} \left| \hat{G}_{n,U}(x) - \left\{ \Phi(x/\hat{\sigma}_n) - \frac{1}{6\sqrt{n}} \frac{\hat{\mu}_{3,n}}{\hat{\sigma}_n^3} \left(\left(\frac{x}{\hat{\sigma}_n}\right)^2 - 1 \right) \phi\left(\frac{x}{\hat{\sigma}_n}\right) \right\} \right| = O_p(n^{-1})$$

as $n \to \infty$, where $\hat{\mu}_{3,n} = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^3$.

From the above, we have

$$\sup_{x \in \mathbb{R}} \left| \hat{G}_{n,U}(x) - G_{n,U}(x) \right| \leq O_p(n^{-1}) + \sup_{x \in \mathbb{R}} \left| \Phi\left(\frac{x}{\sigma}\right) - \Phi\left(\frac{x}{\hat{\sigma}_n}\right) \right|$$

$$+ \frac{1}{6\sqrt{n}} \frac{\hat{\mu}_{3,n}}{\hat{\sigma}_n^3} \sup_{x \in \mathbb{R}} \left| \left[\left(\frac{x}{\hat{\sigma}_n}\right)^2 - 1 \right] \phi\left(\frac{x}{\hat{\sigma}_n}\right) \right|$$

$$+ \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma} \sup_{x \in \mathbb{R}} \left| \left[\left(\frac{x}{\sigma}\right)^2 - 1 \right] \phi\left(\frac{x}{\sigma}\right) \right|$$

$$= O_p(n^{-1/2}),$$

as $n \to \infty$, where the last equality requires a little more work.

For (ii), we have

$$\sup_{x \in \mathbb{R}} \left| G_n(x) - \left\{ \Phi(x) - \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3} (x^2 - 1) \phi(x) \right\} \right| = O(n^{-1})$$

and it can be shown that

$$\sup_{x \in \mathbb{R}} \left| \hat{G}_n(x) - \left\{ \Phi(x) - \frac{1}{6\sqrt{n}} \frac{\hat{\mu}_{3,n}}{\hat{\sigma}_n^3} (x^2 - 1) \phi(x) \right\} \right| = O_p(n^{-1})$$

as $n \to \infty$.

Then we may write

$$\sup_{x \in \mathbb{R}} \left| \hat{G}_n(x) - G_n(x) \right| \leq O_p(n^{-1}) + \frac{1}{6\sqrt{n}} \left| \frac{\hat{\mu}_{3,n}}{\hat{\sigma}_n^3} - \frac{\mu_3}{\sigma^3} \right| \sup_{x \in \mathbb{R}} \left| (x^2 - 1) \phi(x) \right|$$

$$= o_p(n^{-1/2}),$$

since $\hat{\mu}_{3,n} \to \mu_3$ w.p. 1 and $\hat{\sigma}_n^3 \to \sigma^3$ w.p. 1.

For (iii), we have

$$\sup_{x \in \mathbb{R}} \left| G_{n,S}(x) - \left\{ \Phi(x) + \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3} (2x^2+1) \phi(x) \right\} \right| = O(n^{-1})$$

and it can be shown that

$$\sup_{x \in \mathbb{R}} \left| \hat{G}_{n,S}(x) - \left\{ \Phi(x) - \frac{1}{6\sqrt{n}} \frac{\hat{\mu}_{3,n}}{\hat{\sigma}_n^3} (2x^2+1) \phi(x) \right\} \right| = O_p(n^{-1})$$

as $n \to \infty$.

Then we have

$$\sup_{x \in \mathbb{R}} \left| \hat{G}_{n,S}(x) - G_{n,S}(x) \right| \leq O_p(n^{-1}) + \frac{1}{6\sqrt{n}} \left| \frac{\hat{\mu}_{3,n}}{\hat{\sigma}_n^3} - \frac{\mu_3}{\sigma^3} \right| \sup_{x \in \mathbb{R}} \left| [2x^2+1] \phi(x) \right|$$

$$= o_p(n^{-1/2}),$$

using again $\hat{\mu}_{3,n} \to \mu_3$ w.p. 1 and $\hat{\sigma}_n^3 \to \sigma^3$ w.p. 1.